Computer Supported Modeling and Reasoning

David Basin, Achim D. Brucker, Jan-Georg Smaus, and Burkhart Wolff

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Higher-Order Logic: Arithmetic

Burkhart Wolff

The Roadmap

We are still looking at how the different parts of mathematics are encoded in the Isabelle/HOL library.

- Orders
- Sets
- Functions
- (Least) fixpoints and induction
- (Well-founded) recursion
- Arithmetic
- Datatypes

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Current stage of our course:

- On the basis of conservative embeddings, set theory can be built safely.
- Inductive sets can be defined using least fixpoints and suitably supported by Isabelle.
- Well-founded orderings can be defined without referring to infinity. Recursive functions can be based on these. Needs inductive sets though. Support by Isabelle provided.

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Next important topic: arithmetic.

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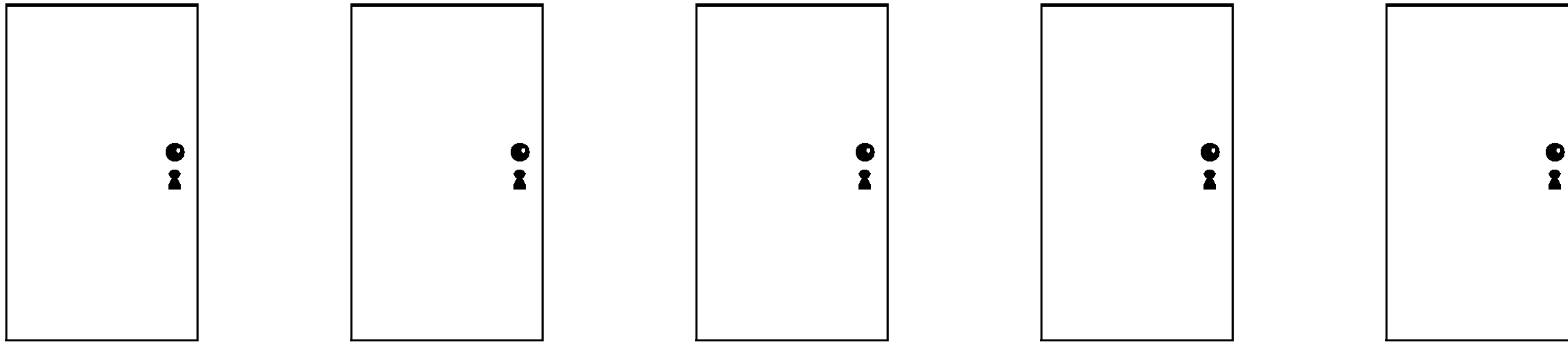
  Yes. Finally use infinity axiom.

What is Infinity? Cantor's Hotel



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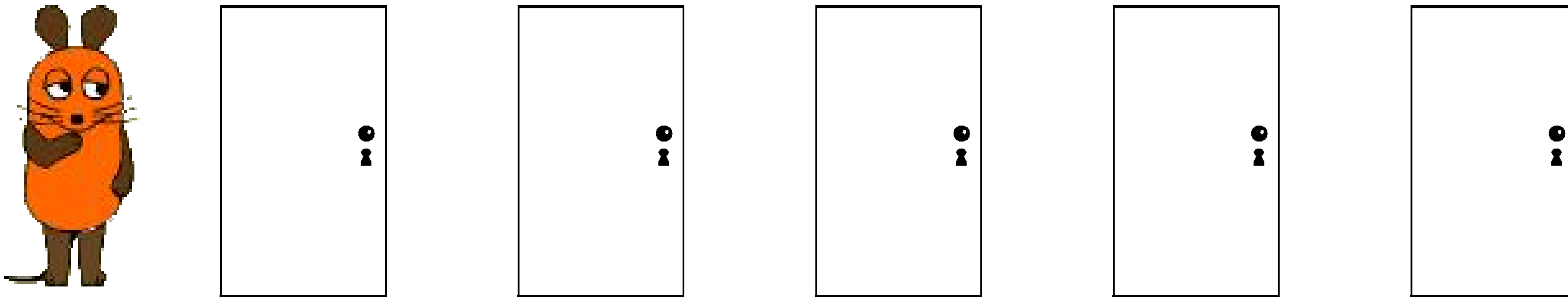

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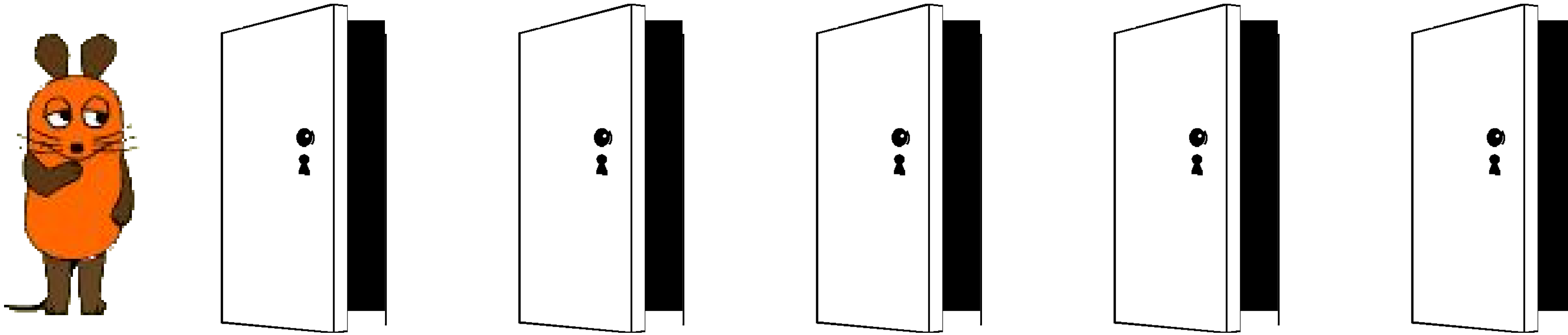

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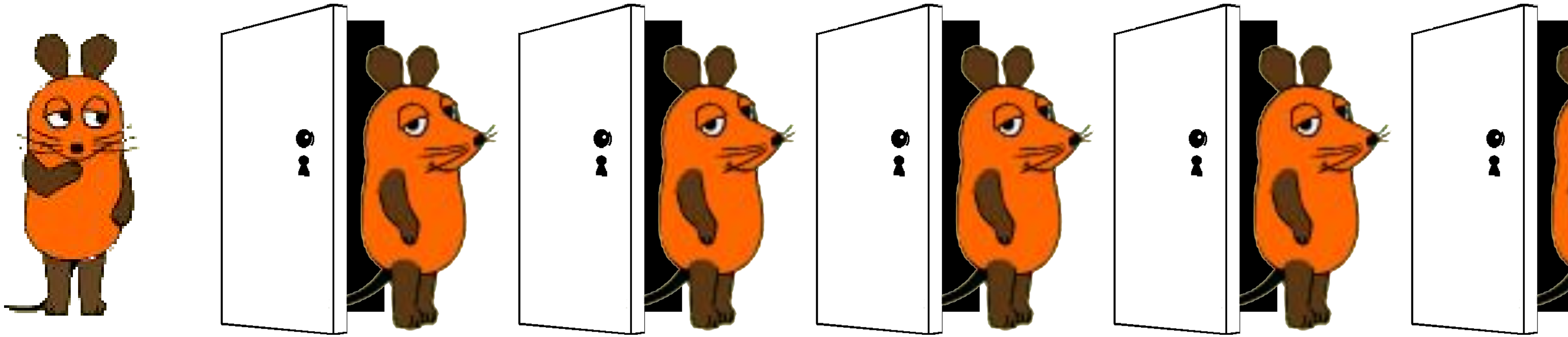

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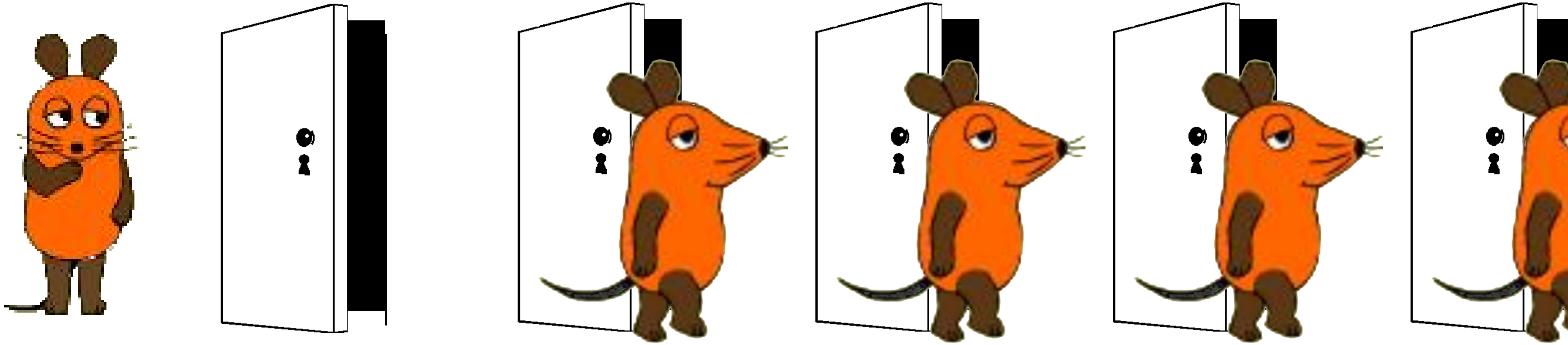

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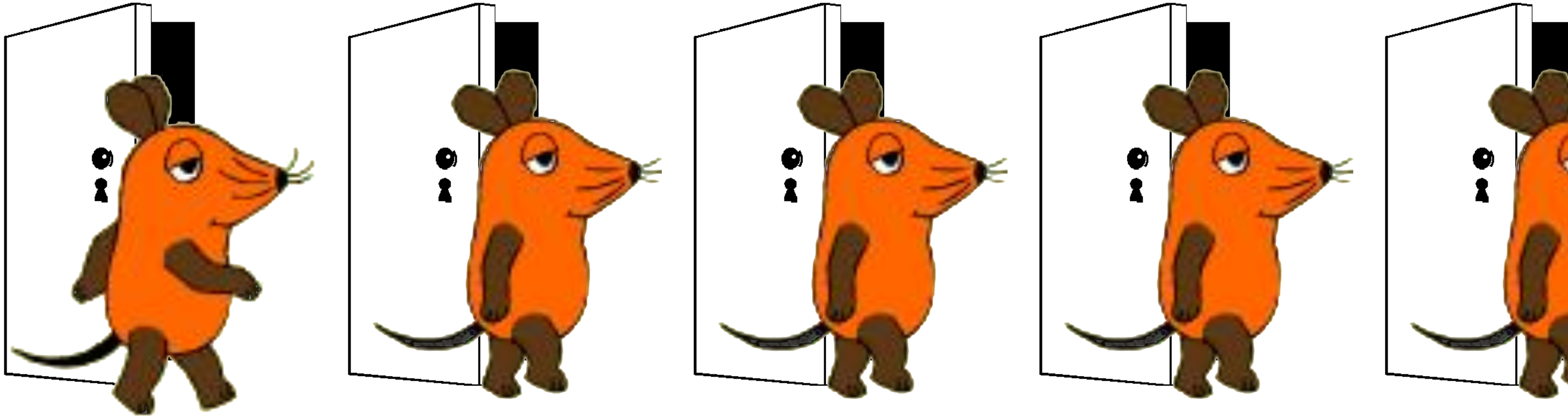

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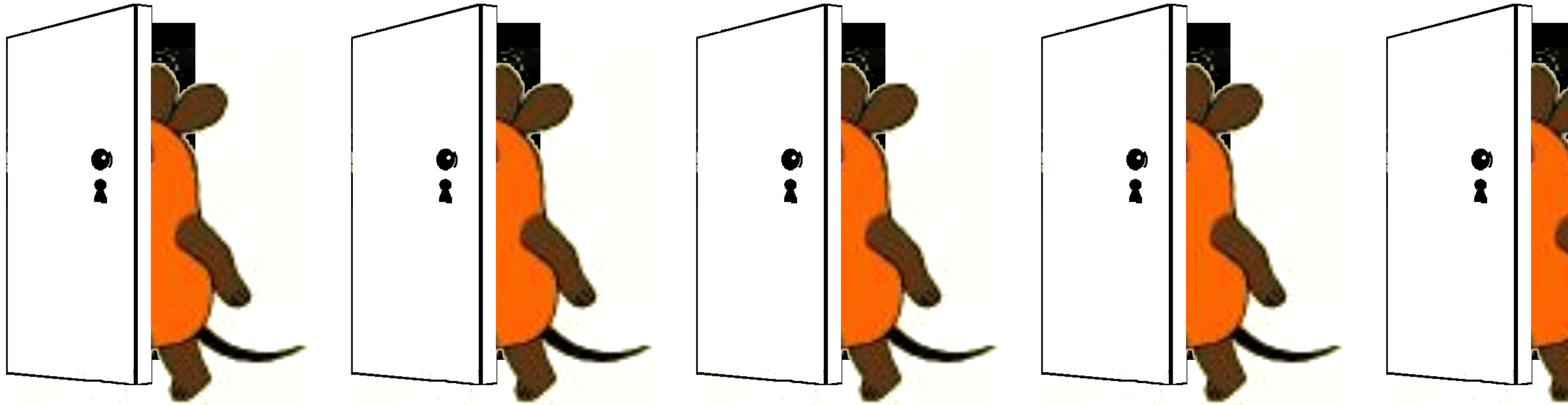

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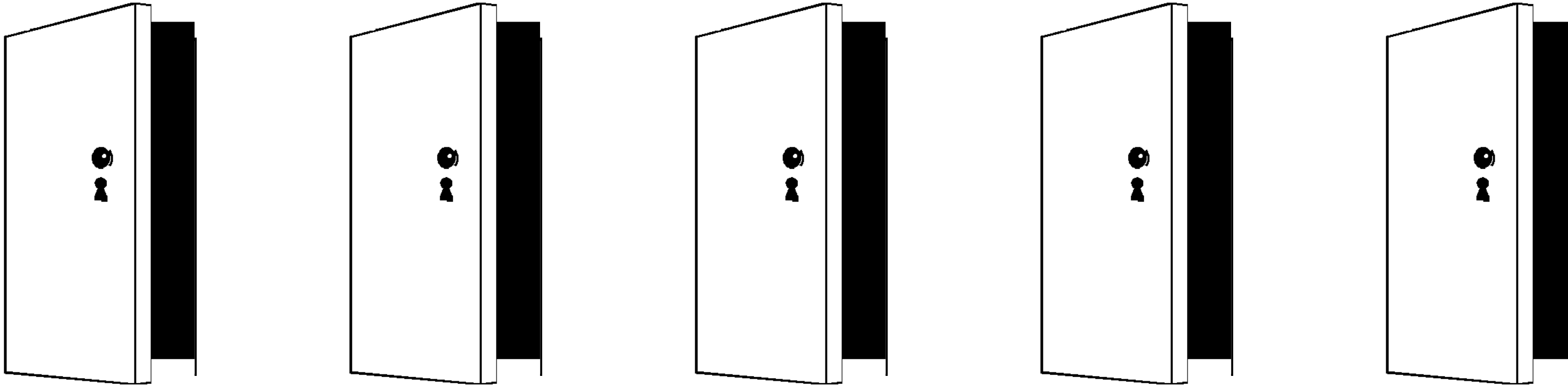

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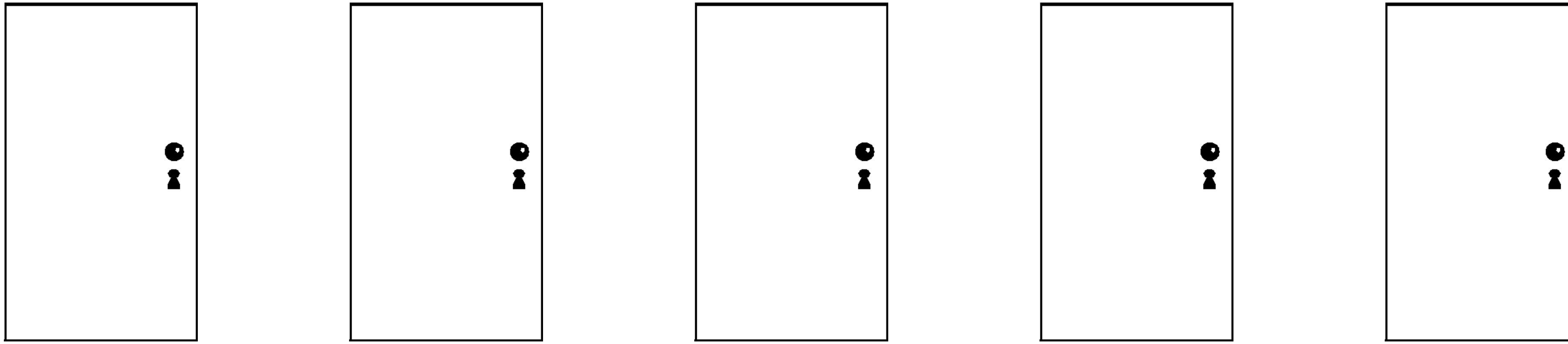

Cantor's hotel has infinitely many guests in his rooms if the receptionist can do the following procedure: A new guest arrives. The receptionist tells all guests to move one room. They move one room forward, the new guest takes the first room, and all are home and dry !

Axiom of Infinity

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The axiom forces ind to be the "infinite type" (called "I" in [Chu40]).

Natural Numbers: Nat.thy

Based on the axiom of inifinity, a *proto*-Zero and a *proto*-Suc can be introduced by type specification:

consts

ZERO :: ind

SUC :: ind $\Rightarrow$ ind

specification (SUC)

SUC_charn: inj SUC $\wedge \neg$ surj SUC

by (*rule* infinity)

specification (ZERO)

ZERO_charn: ZERO $\neq$SUC X

by (*insert* SUC_charn, auto simp: surj_def)

The proofs show that witnesses satisfy the required properties of the constants.

Defining the Set Nat

Now we define inductively a set generated by ZERO and SUC:

consts NAT :: ind set

inductive NAT
  intros
  ZERO_I: ZERO∈NAT
  SUC_I : ⟦ x∈NAT ⟧⟹SUC x∈NAT

(Recall that Isabelle converts this in:
$Nat = lfp\,(\lambda X.\{Zero\_Rep\} \cup (Suc\_Rep\,`\,X))$
and derives an induction scheme)

Defining the Type nat

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typedef (Nat)
  nat = "Nat" **by** (...)

Constants in nat

Moreover, we define 0 and Suc via their corresponding values in Nat :

consts

Suc :: nat ⇒ nat

pred_nat :: (nat × nat) set

defs

Zero_nat_def : 0 ≡ Abs_Nat Zero_Rep

Suc_def : Suc ≡ (λn. Abs_Nat (Suc_Rep (Rep_Nat n)))

pred_nat_def : pred_nat ≡ {(m, n). n = Suc m}

Some Theorems in Nat

From the induction inherited from Nat, we derive:

nat_induct $[\![P\ 0;\ \bigwedge n. P\ n \Longrightarrow P\ (Suc\ n)\ ]\!] \Longrightarrow P\ n$

diff_induct $[\![\bigwedge x.\ P\ x\ 0;\ \bigwedge y.\ P\ 0\ (Suc\ y);$
$\bigwedge x\ y. P\ x\ y \Longrightarrow P\ (Suc\ x)(Suc\ y)]\!]$
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Moreover, we have as pre-requisite for wf-induction:

$$wf(pred\_nat)$$

These are the main weapons for proving theorems in basic number theory.

Nat.thy and Well-Founded Orders

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$$m < n \equiv (m, n) \in pred\_nat\hat{}+$$
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have the properties:

$$m \leq m$$
$$[\![x \leq y; y \leq z]\!] \Longrightarrow x \leq z$$
$$[\![x \leq y; y \leq x]\!] \Longrightarrow x = y$$
$$x < y \vee y < x \vee x = y$$

Using Primitive Recursion

Nat.thy defines rich theory on nat. Uses **primrec** syntax for defining recursive functions, and case construct.

primrec

add_0 $0 + n = n$

add_Suc Suc m + n = Suc(m + n)

primrec

diff_0 $m - 0 = m$

diff_Suc m − Suc n = (case m − n **of** 0 => 0 | Suc k => k)

primrec

mult_0 $0 * n = 0$

mult_Suc Suc m ∗ n = n + (m ∗ n)

Some Theorems in Nat.thy

add_0_right	$m + 0 = m$
add_ac	$m + n + k = m + (n + k)$
	$m + n = n + m$
	$x + (y + z) = y + (x + z)$
mult_ac	$m * n * k = m * (n * k)$
	$m * n = n * m$
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Technically, add_ac and mult_ac are lists of `thm`'s.

Proof of add_0_right

$$m + 0 = m \qquad add\_0\_right$$

Proof of add_0_right

$$n + 0 = n$$

$$\dfrac{\dfrac{\text{add\_0}}{0 + 0 = 0} \qquad Suc\, n + 0 = Suc\, n}{m + 0 = m}\ \text{nat\_induct}$$

Proof of add_0_right

$$\dfrac{\dfrac{\text{add\_0}}{0+0=0} \qquad \dfrac{\dfrac{\dfrac{\text{add\_suc}}{Suc\, m+n = Suc(m+n)}}{Suc(m+n) = Suc\, m+n}\,\textit{sym} \qquad \dfrac{n+0=n}{Suc(n+0) = Suc\, n}\,\textit{fun\_cong}}{Suc\, n+0 = Suc\, n}\,\textit{subst}}{m+0=m}\,\text{nat\_induct}$$

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Integers

The integers ..., $-2, -1, 0, 1, 2,$ ... are identified with equivalence classes over nat $\times$ nat (thought as “differences” $0 - 1$,$1 - 2$,$3 - 4$,...).

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IntDef = Equiv + NatArith +

constdefs

```
   intrel  ::  ((nat×nat) ×(nat ×nat)) set
   intrel  ≡{p. ∃ x1 y1 x2 y2.
                              p=((x1::nat,y1),(x2,y2)) ∧
                              x1+y2 = x2+y1}
```

typedef (Integ)

```
  int = UNIV//intrel  (...)
```

Injections of nat's into integers, negation, addition, multiplication were now defined in terms of "differences":

int :: nat => int

int m ≡Abs_Integ(intrel " {(m,0)})

minus_int_def :

$-$ z ≡Abs_Integ ($\bigcup$(x,y)∈Rep_Integ z. intrel "{(y,x)})

add_int_def :

z $+$ w ≡...

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add_int_def : z ∗ w ≡ ...

Note that we use overloading here!!!

Some Theorems in IntArith

Some theorems on integers are:

zminus_zadd_distrib	$-(z + w) = -z + -w$
zminus_zminus	$-(-z) = z$
zadd_ac	$z1 + z2 + z3 = z1 + (z2 + z3)$
	$z + w = w + z$
	$x + (y + z) = y + (x + z)$
zmult_ac	$z1 * z2 * z3 = z1 * (z2 * z3)$
	$z * w = w * z$
	$z1 * (z2 * z3) = z2 * (z1 * z3)$

Compare to nat theorems.

Further Number Theories

- Binary Integers (Bin.thy, for fast computation)
- Rational Numbers (HOL-Complex/Rational.thy)
- Real Numbers (HOL-Complex/Real.thy: based on Dedekind-sections of positive rationals.
- Hyperreals (HOL-Complex/Hyperreal.thy for non-standard analysis)
- Machine numbers such as JavaIntegers [RW04] and floats [Har98, Har00] for Intel's PentiumIV

Conclusion on Arithmetic

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Potential for

- analysis of processor arithmetic units, and
- function analysis in HOL (combination with computer algebra systems such as Mathematica).

Future: Analysis of hybrid systems.

The methodological overhead of the conservative method can be tackled by powerful mechanical support.

More Detailed Explanations

The Peano Axioms

The Peano axioms are:

- $0 \in nat$
- $\forall x. x \in nat \rightarrow Suc(x) \in nat$
- $\forall x. Suc(x) \neq 0$
- $\forall xy. Suc(x) = Suc(y) \rightarrow x = y$
- $\forall P. (P(0) \wedge \forall n. (P(n) \rightarrow P(Suc(n)))) \rightarrow \forall n. P(n)$

The latter formula is not an axiom in first-order logic, it is traditionally described as “axiom schema”.

However, it fit’s smoothely into HOL.

The case Statement for nat

The case statement for nat is a function of type nat ⇒ nat ⇒ nat) ⇒ nat ⇒ nat. case z f n is defined as follows (using a common mathematical notation):

$$\texttt{case}\ z\ f\ n = \begin{cases} z & \text{if } n = 0 \\ f\ k & \text{if } n = Suc\ k \end{cases}$$

An ML-like pattern match construct in:

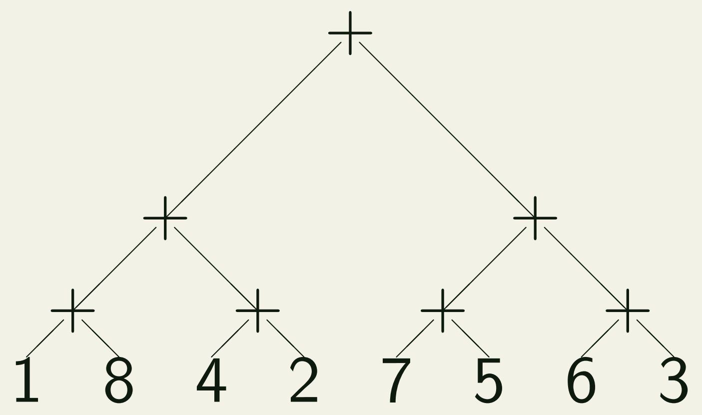
diff_Suc "m − Suc n = (case m − n **of** 0 => 0 | Suc k => k)"

uses a paraphrasing for case 0 (λ x.x) (n−m).

Left Commutation

The theorems $x + (y + z) = y + (x + z)$ and $x * (y * z) = y * (x * z)$ are called left-commutation laws and are crucial for (ordered) rewriting.

Suppose we have the term shown below.



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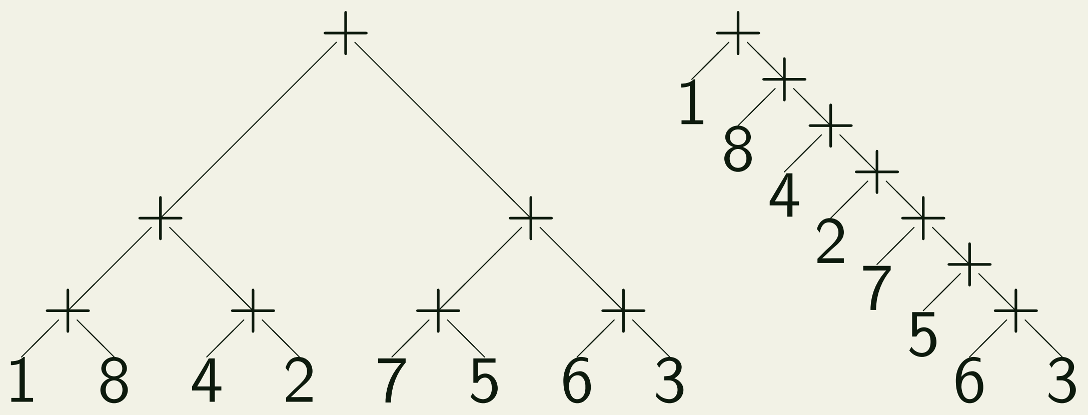

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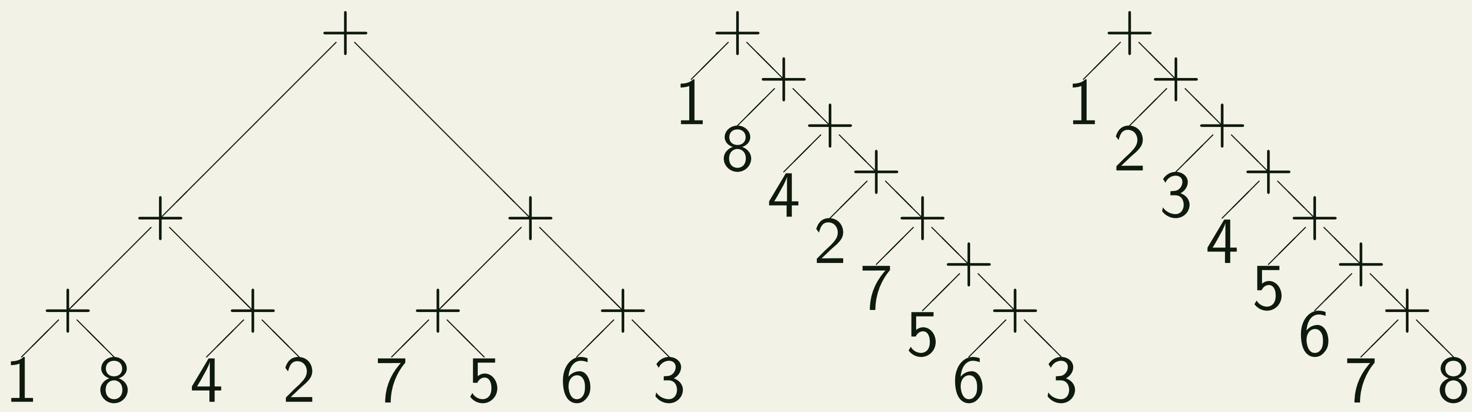
Suppose we have the term shown below. Using associativity ($m + n + k = m + (n + k)$) this will be rewritten to the second term. Using left-commutation, this will be rewritten to the third term. This is a so-called AC-normal form, for an appropriately chosen term ordering.



Equivalence Classes

Recall the general concept of an equivalence relation. Generally, for a set S and an equivalence relation R defined on the set, one can define $S//R$, the quotient of S w.r.t. R.

$$S//R = \{A \mid A \subseteq S \land \forall x, y \in A.(x, y) \in R\}$$

That is, one partitions the set S into subsets such that each subset collects equivalent elements. This is a mathematical standard concept.

We explain it for integers in more detail. One can view a pair (n, m) of natural numbers as representation of the integer $n - m$. But then (n, m) and (n', m') represent the same integer if and only if $n - m = n' - m'$, or equivalently, $n + m' = n' + m$. In this case (n, m) and (n', m') are said to be equivalent. The set of equivalent elements is an equivalence class. The quotient maps therefore a set to a set of equivalence classes.

Reals According to Dedekind

The reals have been axiomatized by Dedekind by stating that a set R is partitioned into two sets A and B such that $R = A \cup B$ and for all $a \in A$ and $b \in B$, we have $a < b$. Now there is a number s such that $a \leq s \leq b$ for all $a \in A$ and $b \in B$. The irrational numbers are characterised by the fact that there exists exactly one such s. This axiomatization has been used as a basis for formalizing real numbers in Isabelle/HOL.

Hyperreals

In non-standard analysis, one works with sequences that are not necessarily converging. This is a relatively new field in mathematics and Isabelle/HOL has been successfully applied in it [FP98]. We just mention this here to say that Isabelle/HOL is used for "cutting-edge" mathematics and not just toy examples.

Hybrid Systems

Hybrid systems is a field in software engineering concerned with using finite automata for controlling physical systems such as ABS in cars etc.

References

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