

Computer Supported Modeling and Reasoning

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Burkhardt Wolff

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<http://www.infsec.ethz.ch/education/permanent/csmr/>

Higher-Order Logic Applications: IMP

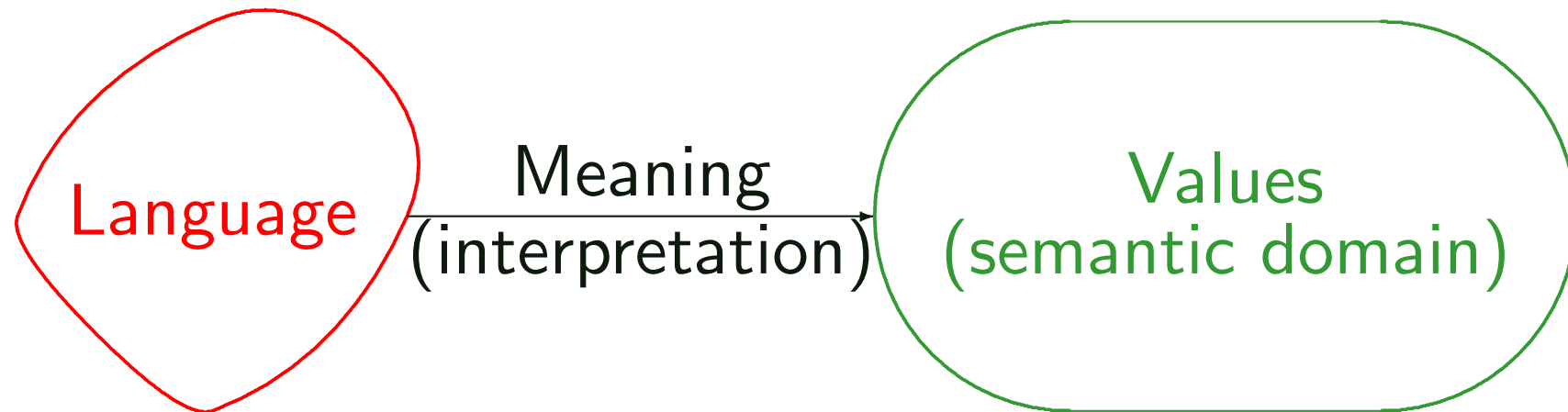
Jan Smaus and Burkhart Wolff

Language Semantics: An Introduction

Question: What is the **meaning** of a (programming or specification) language?

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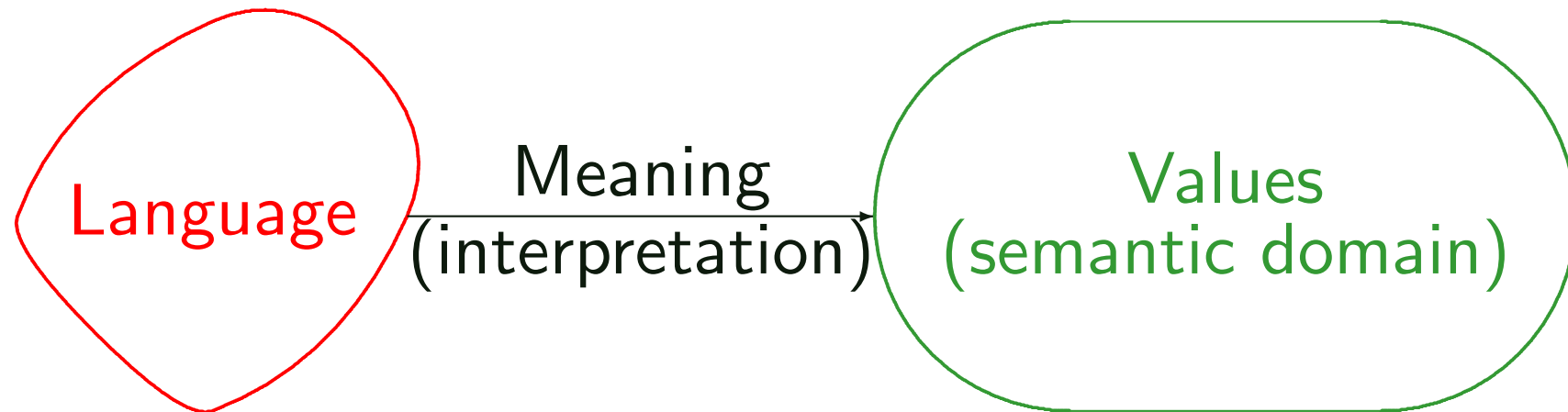
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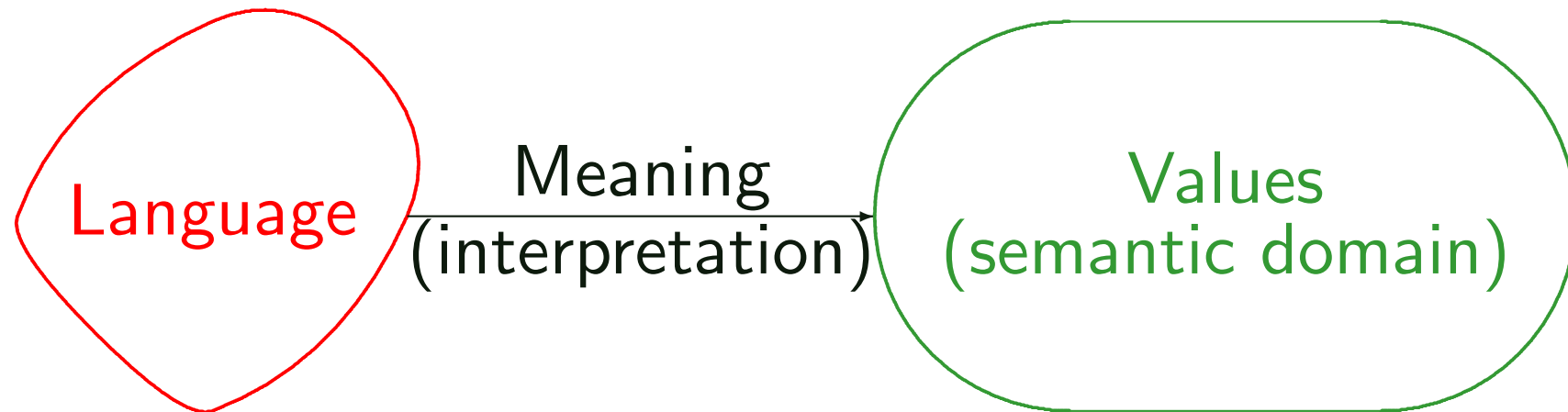
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- Semantics: set of denotations, the **semantic domain**

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- Syntax: language = set of symbols
- Semantics: set of denotations, the **semantic domain**
- **Meaning** is a function (**interpretation** *Sem*) relating them.

Terminology

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- a **shallow embedding** just provides the semantic functions for the operations of the language (no own syntax; e.g. variables are represented by meta-language variables).

Imperative Languages in the Isabelle/HOL Library

There are several embeddings of imperative languages in Isabelle/HOL [Nip02]:

- Hoare
- IMP
- IMPP
- MicroJava

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We choose IMP to learn a bit about “good ole imperative languages”.

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It closely follows the standard textbook [Win96].

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Com = Main +

typedecl loc

types

val = nat (**arb.**)

state = loc \Rightarrow val

aexp = state \Rightarrow val

bexp = state \Rightarrow bool

datatype com =

SKIP

| " ::= " loc aexp (**infixl** 60)

| Semi com com (" _ ; _" [60, 60]10)

| Cond bexp com com

 (" IF _ THEN _ ELSE _" 60)

| While bexp com (" WHILE _ DO _" 60)

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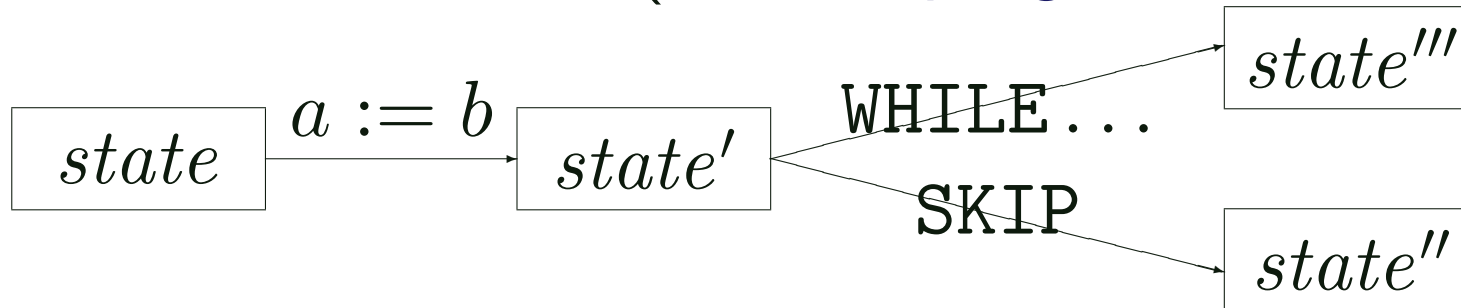
The type `loc` stands for **locations**. Note expressions are represented using the shallow technique. The datatype `com` stands for commands (command sequences).

Operational Semantics: Two Kinds

Natural semantics

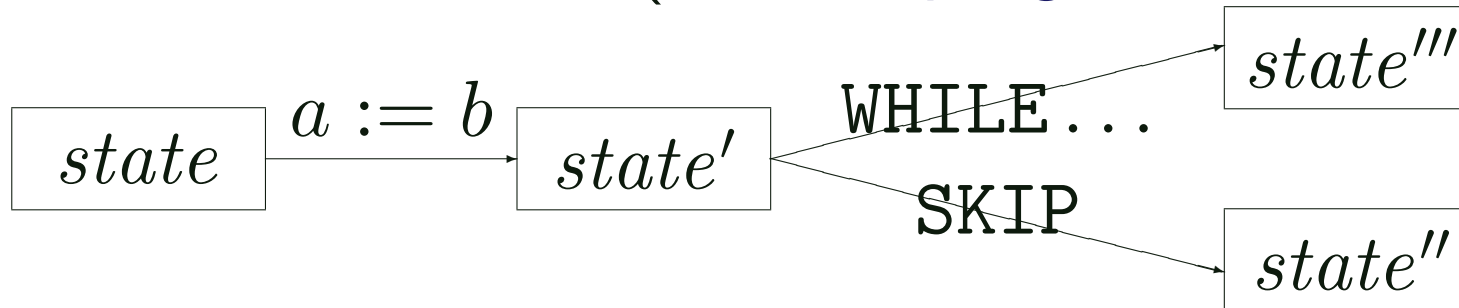
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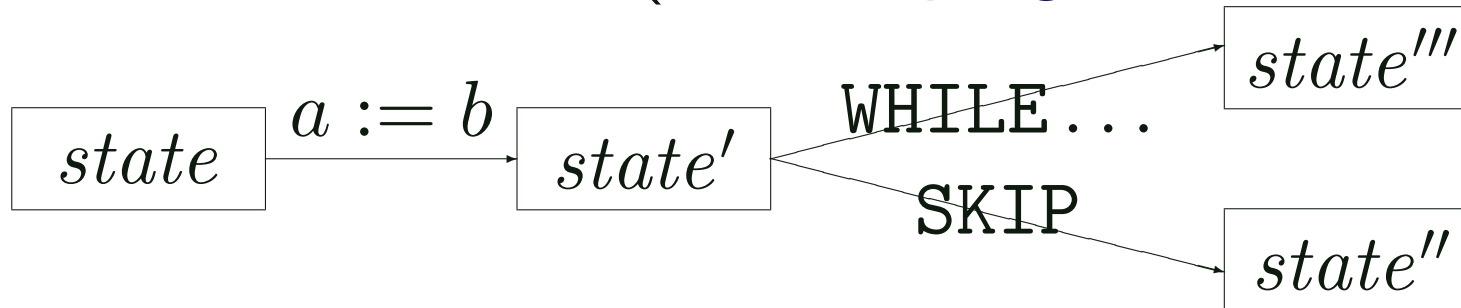
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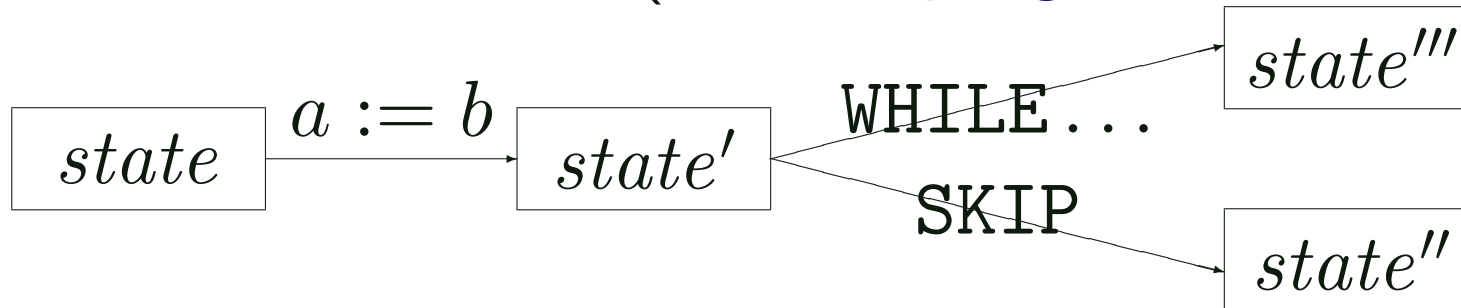


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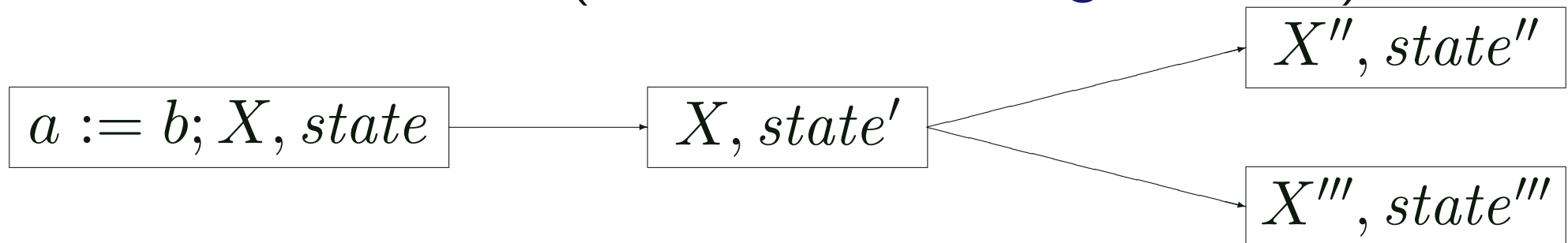
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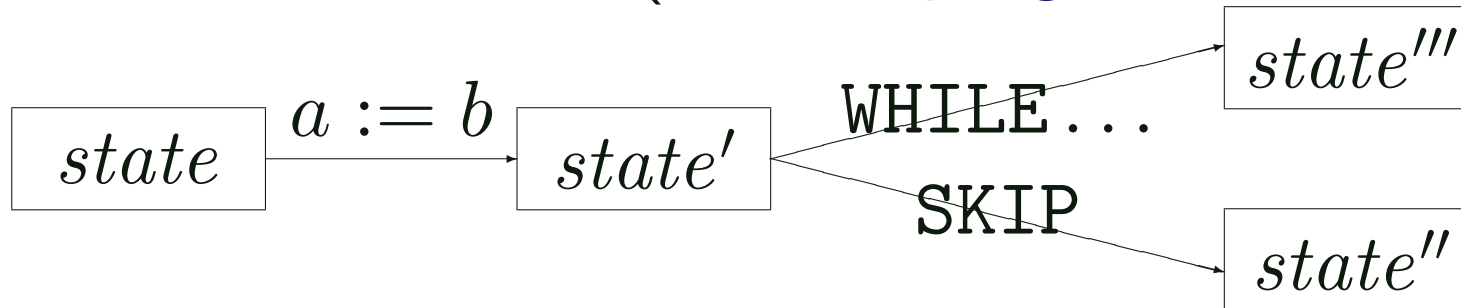
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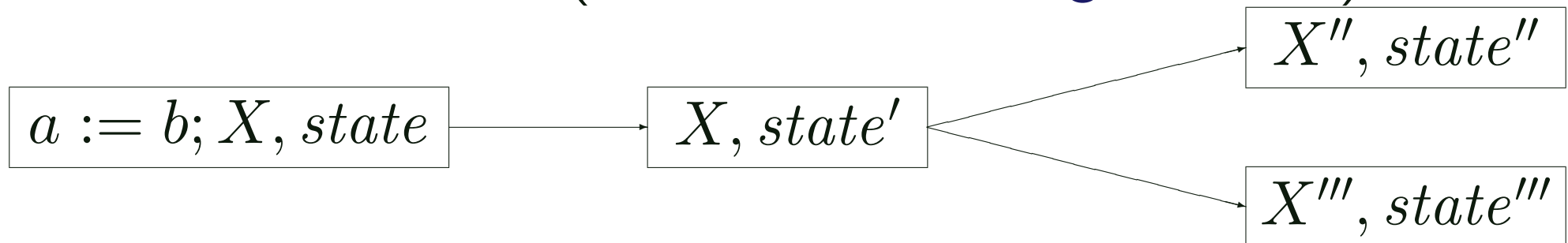
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$evalc1 :: ((\text{com} \times \text{state}) \times (\text{com} \times \text{state})) \text{ set}$

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We now start giving the actual inductive definition of the natural semantics transition relation . . .

Natural Semantics: Skip and Assignment

inductive evalc

intrs

Skip: $\langle \text{SKIP}, s \rangle \text{---c---} s$

Assign $\langle x ::= a, s \rangle \text{---c---} s[x ::= (a \ s)]$

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Note that $s[x ::= (a \ s)]$ is an abbreviation for **update** $s \ x \ (a \ s)$, where

$\text{update } s \ x \ v \equiv \lambda y. \text{ if } y=x \text{ then } v \text{ else } s \ y$

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Note that a is of type aexp or bexp .

Natural Semantics: Sequential Composition

$$\text{Semi } \llbracket \langle c_0, s \rangle \text{ --c--> } s_1; \langle c_1, s_1 \rangle \text{ --c--> } s_2 \rrbracket \\ \implies \langle c_0; c_1, s \rangle \text{ --c--> } s_2$$

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Rationale of natural semantics:

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Rationale of natural semantics:

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- . . . then “jump” via c_1 from s_1 to s_2

Natural Semantics: Control Statements

IfTrue $\llbracket b \ s; \langle c_0, s \rangle \text{---} c \text{---} \rangle s_1 \rrbracket$
 $\implies \langle \text{IF } b \ \text{THEN } c_0 \ \text{ELSE } c_1, s \rangle \text{---} c \text{---} \rangle s_1$

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Note that for non-terminating programs no final state can be derived !

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Transition Semantics: Assignment, Sequential Composition

inductive evalc1

intro

Assign $(x ::= a, s) \text{ --1--> } (\text{SKIP}, s[x ::= a \ s])$

Semi1 $(\text{SKIP}; c, s) \text{ --1--> } (c, s)$

Semi2 $(c_0, s) \text{ --1--> } (c_2, s_1)$

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- transition semantics is close to an abstract machine.

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IfFalse

$$\neg b \ s \Longrightarrow (\text{ IF } \ b \ \text{ THEN } \ c_1 \ \text{ ELSE } \ c_2, s) \text{ --1--} > (c_2, s)$$

WhileFalse

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A non-terminating loop always leads to successor configurations . . .

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Equivalence

Natural semantics vs. transition semantics.

Theorem 1 (`evalc1_eq_evalc`):

$$(c, s) \text{ --*--> } (\text{SKIP}, t) = (\langle c, s \rangle \text{ --c--> } t)$$

Proof: by induction over com.

Denotational Semantics

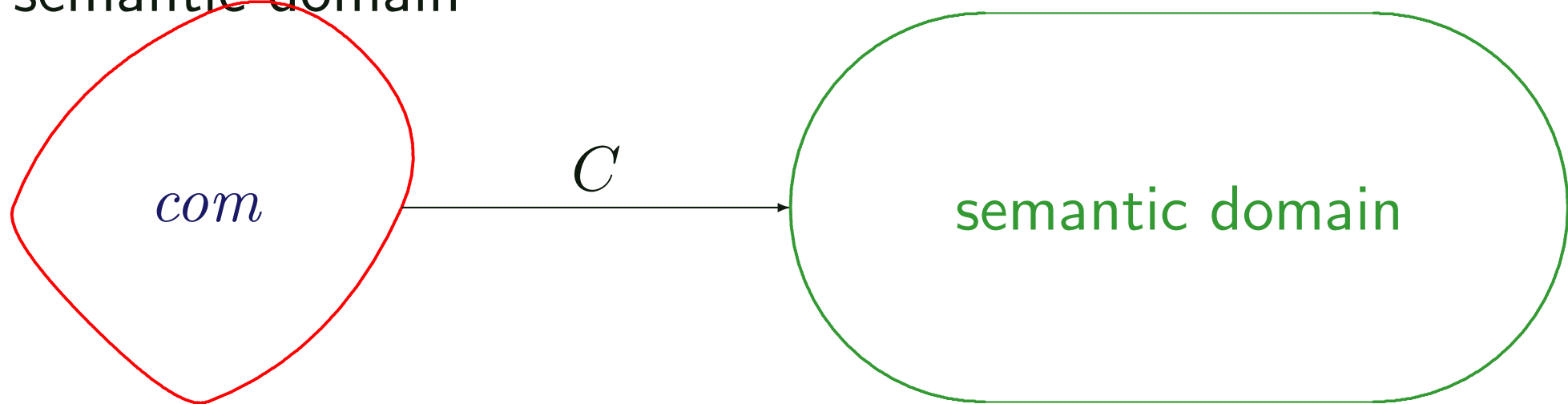
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Denotational Semantics

Idea: Explain recursion as fixpoint construction on the semantic domain

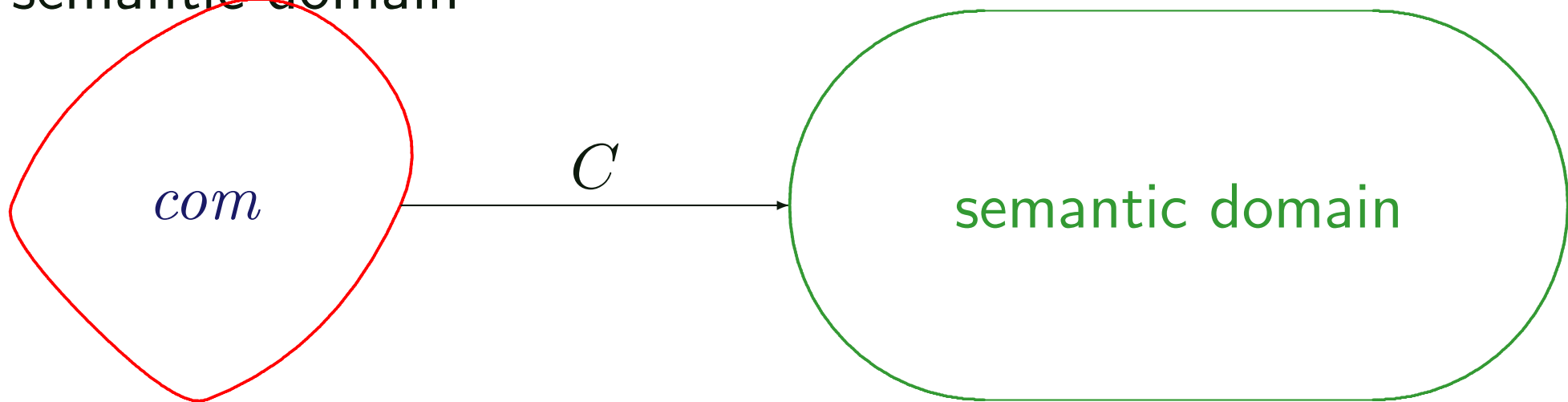
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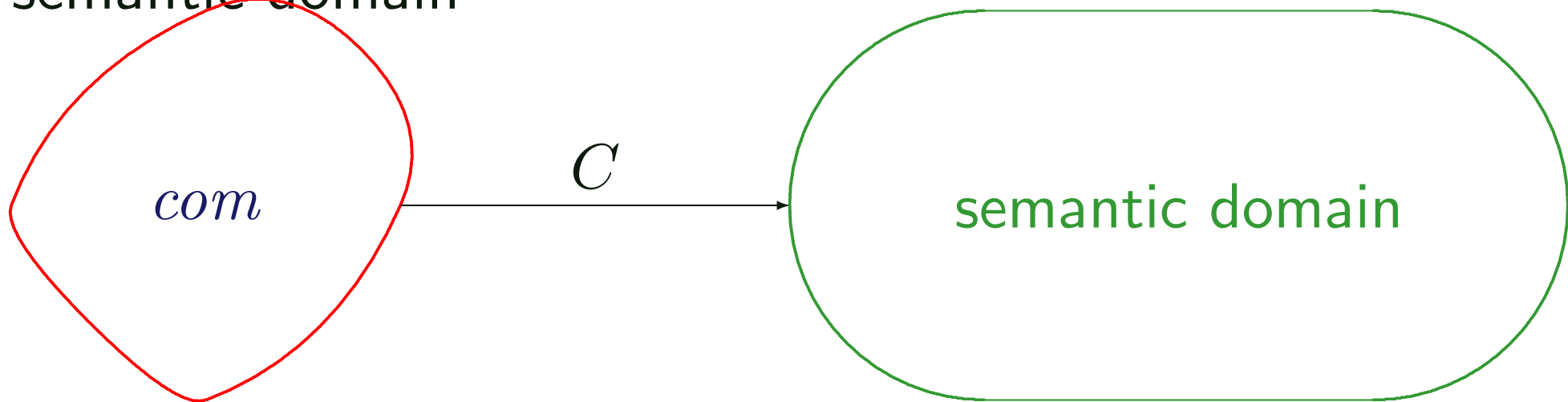


An (imperative) semantic domain is a state relation:

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Semantic function:

consts $C :: \text{com} \Rightarrow \text{com_den}$

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$$C_skip \quad C(\text{SKIP}) = \text{Id}$$

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$$C_comp \quad C(c_0 ; c_1) = C(c_1) \circ C(c_0)$$

$$C_if \quad C(\text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2) = \\ \{(s,t). (s,t) \in C(c_1) \wedge b(s)\} \cup \\ \{(s,t). (s,t) \in C(c_2) \wedge \neg b(s)\}$$

$$C_while \quad C(\text{WHILE } b \text{ DO } c) = \text{lfp } (\text{Gamma } b \text{ (} C \text{ } c))$$

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$$\text{Gamma } b \text{ } cd \equiv (\lambda \text{phi}. \{(s,t). (s,t) \in (\text{phi} \circ cd) \wedge b(s)\} \cup \\ \{(s,t). s=t \wedge \neg b(s)\})$$

Equivalence of Programs

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Theorem 2 (C_While_If):

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Such results justify **program transformations**.

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Theorem 3 (denotational is natural):

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Theorem 3 (denotational is natural):

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Still, if we want to prove **properties of states** to hold, we need different proof techniques. An answer to this need is **axiomatic semantics** or **Hoare Logics**.

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Hoare Logics

The key concept of a Hoare Logics is a **Hoare Triple**

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The “logic” itself is an inductive definition: . . .

Hoare Logics: SKIP

inductive hoare

intro

skip " $\vdash \{P\} \text{SKIP } \{P\}$ "

. . .

Hoare Logics: Assignment

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 $a \equiv \lambda s. 1$ and $P \equiv \lambda s. s \ x = 1$

$$\begin{aligned} & \{\lambda s. (\lambda s. s \ x = 1)(s[x ::= 1])\} x ::= \lambda s. 1 \{\lambda s. s \ x = 1\} \longrightarrow_{\beta} \\ & \{\lambda s. (s[x ::= 1])x = 1\} x ::= \lambda s. 1 \{\lambda s. s \ x = 1\} \longrightarrow_{\beta} \\ & \{\lambda s. True\} x ::= \lambda s. 1 \{\lambda s. s \ x = 1\} \end{aligned}$$

Hoare Logics: Sequence,IF

semi $\llbracket \vdash \{P\}c\{Q\}; \vdash \{Q\}d\{R\} \rrbracket$
 $\implies \vdash \{P\} c;d \{R\}$

If $\llbracket \vdash \{\lambda s. P s \wedge b s\}c\{Q\};$
 $\vdash \{\lambda s. P s \wedge \neg b s\}d\{Q\} \rrbracket$
 $\implies \vdash \{P\} \text{ IF } b \text{ THEN } c \text{ ELSE } d \{Q\}$ "

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The rule for IF represents, as expected, the case split.

Hoare Logics: WHILE

While

$$\begin{aligned} & \llbracket \vdash \{ \lambda s. P \ s \ \wedge \ b \ s \} \ c \ \{ P \} \rrbracket \\ & \implies \vdash \{ P \} \text{ WHILE } b \text{ DO } c \{ \lambda s. P \ s \ \wedge \ \neg b \ s \}'' \end{aligned}$$

Hoare Logics: WHILE

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If WHILE terminates, then the contrary of the condition must hold. . .

Hoare Logics: Consequence Rule

conseq $\left[\begin{array}{l} \forall s. P' s \longrightarrow P s; \\ \vdash \{P\}c\{Q\}; \\ \forall s. Q s \longrightarrow Q' s \end{array} \right] \\ \implies \vdash \{P'\}c\{Q'\}$

Hoare Logics: Consequence Rule

$$\text{conseq} \quad \left[\begin{array}{l} \forall s. P' s \longrightarrow P s; \\ \vdash \{P\}c\{Q\}; \\ \forall s. Q s \longrightarrow Q' s \end{array} \right] \\ \implies \vdash \{P'\}c\{Q'\}$$

One can always **strengthen** the pre-condition or **weaken** the post-condition.

Hoare Logics at a Glance

inductive hoare

intro

skip $\vdash \{P\} \text{SKIP} \{P\}$

ass $\vdash \{\lambda s. P(s[x ::= a \ s])\} \ x ::= a \ \{P\}$

semi $\llbracket \vdash \{P\}c\{Q\}; \vdash \{Q\}d\{R\} \rrbracket \implies \vdash \{P\} \ c;d \ \{R\}$

If $\llbracket \vdash \{\lambda s. P \ s \wedge b \ s\}c\{Q\}; \vdash \{\lambda s. P \ s \wedge \neg b \ s\}d\{Q\} \rrbracket$
 $\implies \vdash \{P\} \ \text{IF } b \ \text{THEN } c \ \text{ELSE } d \ \{Q\}$

While $\vdash \{\lambda s. P \ s \wedge b \ s\} \ c \ \{P\}$
 $\implies \vdash \{P\} \ \text{WHILE } b \ \text{DO } c \ \{\lambda s. P \ s \wedge \neg b \ s\}$

conseq $\llbracket \forall s. P' \ s \longrightarrow P \ s; \vdash \{P\}c\{Q\} \rrbracket$

$$\forall s. \llbracket Q \ s \longrightarrow Q' \ s \rrbracket \implies \vdash \{P'\}_c\{Q'\}$$

Validity Relation

We define a **validity relation**:

$$|= \{P\}c\{Q\} \equiv \forall s. \forall t. (s,t) \in C(c) \longrightarrow P s \longrightarrow Q t''$$

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$$|= \{P\}c\{Q\} \equiv \forall s. \forall t. (s,t) \in C(c) \longrightarrow P s \longrightarrow Q t''$$

Validity represents our intuition of what Hoare triples mean: whenever the program c can make a transition from s to t (wrt. to the underlying operational/denotational semantics), and whenever P holds for s , Q must hold for t .

Relating Hoare and Denotational Semantics

Theorem 4 (Hoare soundness):

$$\vdash \{P\} c \{Q\} \implies \models \{P\} c \{Q\}$$

Theorem 5 (Hoare relative completeness):

$$\models \{P\} c \{Q\} \implies \vdash \{P\} c \{Q\}$$

Why relative?

So the Hoare relation is in fact compatible with the denotational semantics of IMP.

Example Program

```
tm   ::= λs. 1;
sum  ::= λs. 1;
i    ::= λs. 0;
WHILE λs. (s sum) <= (s a) DO
  (i   ::= λs. (s i) + 1;
   tm  ::= λs. (s tm) + 2;
   sum ::= λs. (s tm) + (s sum))
```

What does this program do?

Example Program

```
tm   ::=  $\lambda s. 1$ ;
sum  ::=  $\lambda s. 1$ ;
i    ::=  $\lambda s. 0$ ;
WHILE  $\lambda s. (s \text{ sum}) \leq (s \text{ a})$  DO
  (i   ::=  $\lambda s. (s \text{ i}) + 1$ ;
   tm  ::=  $\lambda s. (s \text{ tm}) + 2$ ;
   sum ::=  $\lambda s. (s \text{ tm}) + (s \text{ sum})$ )
```

What does this program do?

Try $a = 1$, $a = 2$, . . . , and look at i !

Square Root

Answer: The program computes the square root. Informally:

$$Pre \equiv \text{” True”}$$

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Formally

$$Pre \equiv \lambda s. True$$

Square Root

Answer: The program computes the square root. Informally:

$$\begin{aligned} Pre &\equiv \text{'' True''} \\ Post &\equiv \text{'' } i^2 \leq a < (i + 1)^2 \text{''} \end{aligned}$$

Formally

$$\begin{aligned} Pre &\equiv \lambda s. \text{ True} \\ Post &\equiv \lambda s. (s\ i) * (s\ i) \leq (s\ a) \wedge \\ &\quad s\ a < (s\ i + 1) * (s\ i + 1) \end{aligned}$$

Proving $\{Pre\} \dots \{Post\}$

When using the Hoare-calculus **directly**:

- we apply the rules following the syntax
- we can apply the conseq-rule initially (interfacing Pre and $Post$)
- we can apply the conseq-rule when entering a **WHILE** (interfacing the invariant)

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Abbreviation: $ExcC \equiv \lambda s. Inv\ s \wedge \neg s\ sum \leq s\ a$ (“exit condition”).

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When using the Hoare-calculus **directly**:

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Abbreviation: $ExC \equiv \lambda s. Inv\ s \wedge \neg s\ sum \leq s\ a$ (“exit condition”). We will develop the proof and from its structure “guess” the **Invariant**.

Proof

$$\{Pre\} \boxed{tm \dots} \{Post\}$$

This is what we want to prove.

Proof

$$\frac{\boxed{} \quad \{ \} \boxed{tm \dots} \{Exc\} \quad \boxed{\mathcal{I}_2}}{\{Pre\} \boxed{tm \dots} \{Post\}} \textit{conseq}$$

Nothing happens **after** the loop, so intuition says that Exc must imply $Post$.

Proof

$$\begin{array}{c}
 \boxed{} \quad \{PW\} \boxed{WH \dots} \{ExC\} \\
 \hline
 \boxed{} \quad \{ } \boxed{i \dots} \{ExC\} \quad \text{semi} \\
 \hline
 \boxed{} \quad \{ } \boxed{sum \dots} \{ExC\} \quad \text{semi} \\
 \hline
 \boxed{} \quad \{ } \boxed{tm \dots} \{ExC\} \quad \boxed{\mathcal{I}_2} \quad \text{semi} \\
 \hline
 \boxed{} \quad \{ } \boxed{tm \dots} \{ExC\} \quad \boxed{\mathcal{I}_2} \quad \text{conseq} \\
 \hline
 \{Pre\} \boxed{tm \dots} \{Post\}
 \end{array}$$

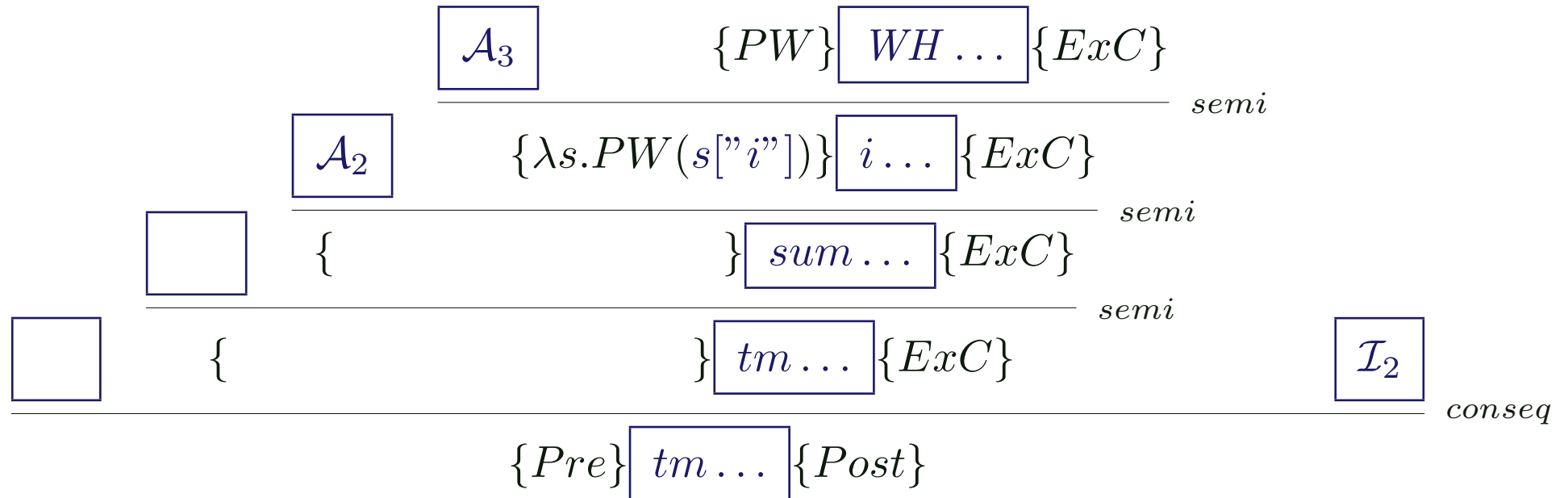
Apply *semi* three times. *PW* (“pre while”) is just a sensible choice of name: we don’t know yet what it is.

Proof

$$\begin{array}{c}
 \boxed{A_3} \quad \{PW\} \boxed{WH \dots} \{ExC\} \\
 \hline \text{semi} \\
 \boxed{} \quad \{ \quad \} \boxed{i \dots} \{ExC\} \\
 \hline \text{semi} \\
 \boxed{} \quad \{ \quad \} \boxed{sum \dots} \{ExC\} \\
 \hline \text{semi} \\
 \boxed{} \quad \{ \quad \} \boxed{tm \dots} \{ExC\} \quad \boxed{I_2} \\
 \hline \text{conseq} \\
 \{Pre\} \boxed{tm \dots} \{Post\}
 \end{array}$$

This application of *ass* will allow us to reconstruct the precondition in the line just below.

Proof



And likewise $\boxed{A_2}$.

Proof

$$\begin{array}{c}
 \boxed{A_3} \quad \{PW\} \boxed{WH \dots} \{ExC\} \\
 \hline
 \boxed{A_2} \quad \{\lambda s.PW(s["i"])\} \boxed{i \dots} \{ExC\} \quad \text{semi} \\
 \hline
 \boxed{A_1} \quad \{\lambda s.PW(s["i, sum"])\} \boxed{sum \dots} \{ExC\} \quad \text{semi} \\
 \hline
 \boxed{} \quad \{ \} \boxed{tm \dots} \{ExC\} \quad \boxed{I_2} \quad \text{semi} \\
 \hline
 \{Pre\} \boxed{tm \dots} \{Post\} \quad \text{conseq}
 \end{array}$$

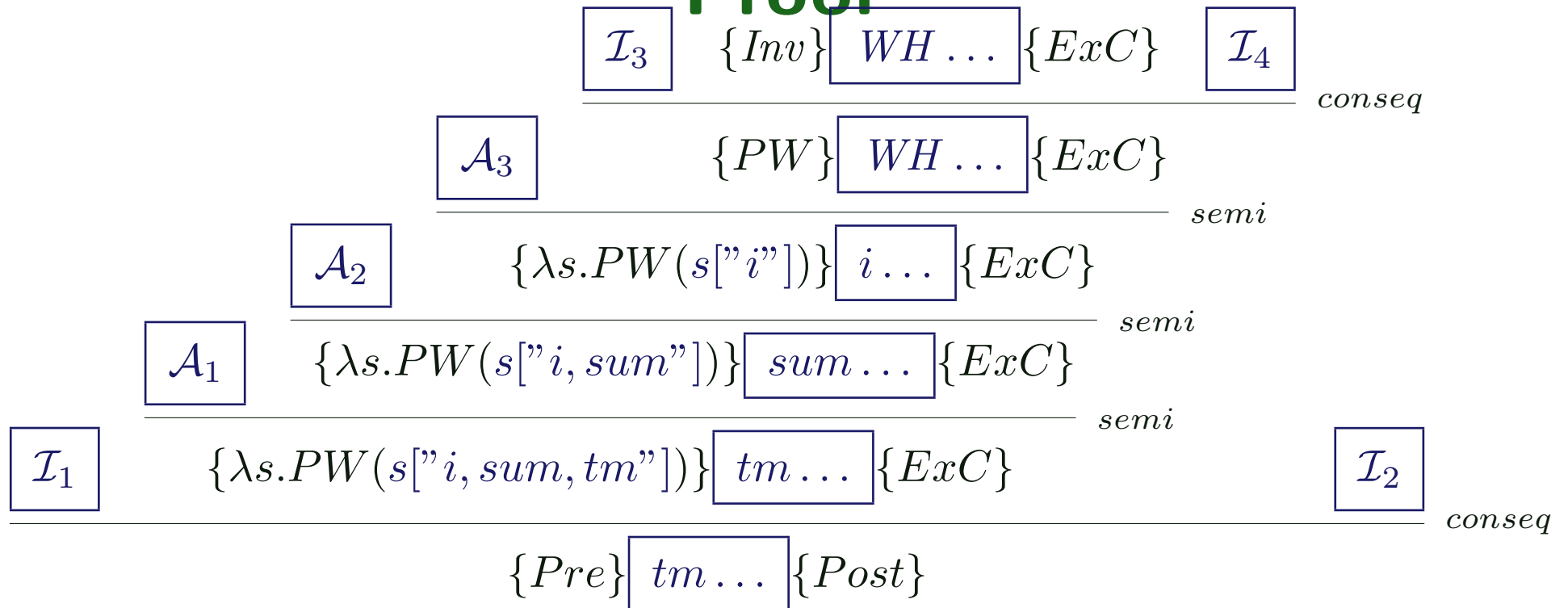
And likewise $\boxed{A_1}$.

Proof

$$\begin{array}{c}
 \boxed{A_3} \quad \{PW\} \boxed{WH \dots} \{ExC\} \\
 \hline
 \boxed{A_2} \quad \{\lambda s.PW(s["i"])\} \boxed{i \dots} \{ExC\} \quad \text{semi} \\
 \hline
 \boxed{A_1} \quad \{\lambda s.PW(s["i, sum"])\} \boxed{sum \dots} \{ExC\} \quad \text{semi} \\
 \hline
 \boxed{I_1} \quad \{\lambda s.PW(s["i, sum, tm"])\} \boxed{tm \dots} \{ExC\} \quad \boxed{I_2} \quad \text{semi} \\
 \hline
 \{Pre\} \boxed{tm \dots} \{Post\} \quad \text{conseq}
 \end{array}$$

We now know (by the form of *conseq*) what $\boxed{I_1}$ is.

Proof



Intuition says that PW must imply Inv .
 Of course, we are not ready yet. . .

Completing the Proof

A_1 , A_2 and A_3 are complete, and I_4 is trivial.

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This also involves the question how the metavariables must be instantiated.

What is PW ?

The metavariable PW (“precondition of $WHILE$ ”) must fulfill (to show $\boxed{\mathcal{I}_1}$)

$$\forall s. Pre\ s \rightarrow PW(s[i ::= 0][sum ::= 1][tm ::= 1])$$

What is PW ?

The metavariable PW (“precondition of $WHILE$ ”) must fulfill (to show $\boxed{\mathcal{I}_1}$)

$$\forall s. Pre\ s \rightarrow PW(s[i ::= 0][sum ::= 1][tm ::= 1])$$

Solution (recall that $Pre \equiv \lambda s. True$):

$$PW = \lambda s. s\ i = 0 \wedge s\ sum = 1 \wedge s\ tm = 1$$

What is *Inv*?

Continuing our proof tree construction:

$$\frac{\{\lambda s. Inv\ s \wedge s\ sum \leq s\ a\} \boxed{\text{"body"}} \{Inv\}}{\{Inv\} \boxed{WH \dots} \{Exc\}} \textit{While}$$

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Continuing our proof tree construction:

$$\begin{array}{c}
 \{\lambda s. Inv\ s \wedge s\ sum \leq s\ a\}i ::= \lambda s. s\ i + 1\{P'\} \\
 \{P'\}tm ::= \lambda s. s\ tm + 2\{P''\} \\
 \{P''\}sum ::= \lambda s. s\ tm + s\ sum\{Inv\} \\
 \hline
 \{\lambda s. Inv\ s \wedge s\ sum \leq s\ a\} \boxed{\text{"body"}} \{Inv\} \quad \text{semi}^2 \\
 \hline
 \{Inv\} \boxed{WH \dots} \{ExC\} \quad \text{While}
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Just blindly applying *semi* twice gives three formulas to be proven using *ass*, one for each assignment in the loop.

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Just blindly applying *semi* twice gives three formulas to be proven using *ass*, one for each assignment in the loop.

Now what are P' and P'' ? Have a look at rule *ass* first!

Calculating P' and P'' (by Rule *ass*)

$$P'' = \lambda s. Inv(s[sum ::= s tm + s sum])$$

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 &\quad (s'[\text{tm} ::= s' \text{ tm} + 2]) \\
 &= \lambda s'. \text{Inv}((s'[\text{tm} ::= s' \text{ tm} + 2]) \\
 &\quad [\text{sum} ::= (s'[\text{tm} ::= s' \text{ tm} + 2]) \text{ tm} + \\
 &\quad \quad (s'[\text{tm} ::= s' \text{ tm} + 2]) \text{ sum}]) \\
 &= \lambda s'. \text{Inv}(s'[\text{tm} ::= s' \text{ tm} + 2] \\
 &\quad [\text{sum} ::= s' \text{ tm} + 2 + s' \text{ sum}]).
 \end{aligned}$$

Applying *ass* to $i ::= \lambda s. s\ i + 1$

Now treat $i ::= \lambda s. s\ i + 1$ in the same way. Temporarily, let's write P for $\lambda s. Inv\ s \wedge s\ sum \leq s\ a$. Recall $P' =$

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Applying *ass* to $i ::= \lambda s. s\ i + 1$

$$\lambda s. Inv\ s \wedge s\ sum \leq s\ a$$

$$= \lambda s. Inv(s[i ::= s\ i + 1][tm ::= s\ tm + 2][sum ::= s\ tm + 2 + s\ sum]).$$

So *Inv* must solve this equation.

Inv Must Fulfill the Equation

Inv must fulfill the equation

$$\begin{aligned} & \lambda s. \textit{Inv} \ s \wedge s \ \textit{sum} \leq s \ a = \\ & \lambda s. \textit{Inv} (s[i ::= s \ i + 1][tm ::= s \ tm + 2] \\ & \quad [sum ::= s \ tm + 2 + s \ sum]) \end{aligned}$$

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We can replace λ by \forall due to extensionality.

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We can replace λ by \forall due to extensionality.

Guessing the right *Inv* is obviously difficult! **Informally**

$$\textit{Inv} \equiv \text{''}(i + 1)^2 = \textit{sum} \wedge tm = (2 * i) + 1 \wedge i^2 \leq a\text{''}$$

Checking that *Inv* Fulfills equation

$$s \text{ sum} \leq s \text{ a} \quad \wedge \quad (1)$$

$$(s \text{ i} + 1)^2 = (s \text{ sum}) \quad \wedge \quad (2)$$

$$s \text{ tm} = (2 * (s \text{ i})) + 1 \quad \wedge \quad (3)$$

$$(s \text{ i})^2 \leq (s \text{ a}) \quad \wedge \quad (4)$$

$$(\text{recall: } = \text{ means } \leftrightarrow) \quad = \quad (5)$$

$$((s \text{ i} + 1) + 1)^2 = (s \text{ sum}) + (s \text{ tm}) + 2 \quad \wedge \quad (6)$$

$$(s \text{ tm} + 2) = (2 * (s \text{ i} + 1)) + 1 \quad \wedge \quad (7)$$

$$(s \text{ i} + 1)^2 \leq (s \text{ a}) \quad (8)$$

Proof Sketch

First show the “ \rightarrow ”-direction:

(3) \rightarrow (7) and (1) \wedge (2) \rightarrow (8) by simple arithmetic. (6) is shown as follows:

$$\begin{aligned} ((s\ i + 1) + 1)^2 &= (s\ i + 1)^2 + 2 * (s\ i + 1) + 1 \\ &\stackrel{(2)}{=} (s\ sum) + 2(s\ i) + 1 + 2 \\ &\stackrel{(3)}{=} (s\ sum) + (s\ tm) + 2 \end{aligned}$$

Proof Sketch (Cont.)

Now show the “ \leftarrow ”-direction:

(7) \rightarrow (3) and (8) \rightarrow (4) by simple arithmetic. (2) is shown as follows:

$$\begin{aligned}(s\ i + 1)^2 &= ((s\ i + 1) + 1)^2 - 2 * (s\ i + 1) - 1 \\ &\stackrel{(6)}{=} (s\ sum) + (s\ tm) + 2 - 2 * (s\ i + 1) - 1 \\ &\stackrel{(7)}{=} (s\ sum) + 2 * (s\ i + 1) + 1 \\ &\quad - 2 * (s\ i + 1) - 1 \\ &= s\ sum\end{aligned}$$

Finally, (2) \wedge (8) \rightarrow (1).

Proof Sketch (Cont.)

Now show the “ \leftarrow ”-direction:

(7) \rightarrow (3) and (8) \rightarrow (4) by simple arithmetic. (2) is shown as follows:

$$\begin{aligned}(s\ i + 1)^2 &= ((s\ i + 1) + 1)^2 - 2 * (s\ i + 1) - 1 \\ &\stackrel{(6)}{=} (s\ sum) + (s\ tm) + 2 - 2 * (s\ i + 1) - 1 \\ &\stackrel{(7)}{=} (s\ sum) + 2 * (s\ i + 1) + 1 \\ &\quad - 2 * (s\ i + 1) - 1 \\ &= s\ sum\end{aligned}$$

Finally, (2) \wedge (8) \rightarrow (1). So *Inv* is indeed an invariant!

The *WHILE* Loop: Remarks

We have shown

$$(\text{"enter condition"} \wedge \text{"invar. at entry"}) \leftrightarrow \text{"invar. at exit"}$$

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We can show this because our invariant is so **strong**: for showing \rightarrow , the **weaker** invariant (2) \wedge (3), i.e.

$$”(i + 1)^2 = sum \wedge tm = (2 * i) + 1$$

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$$”(i + 1)^2 = sum \wedge tm = (2 * i) + 1$$

would do (check it!).

But the extra condition $i^2 \leq a$ is needed for showing *Post*, which states what the program actually computes.

Taking Care of $Post$

We have shown $\boxed{\mathcal{I}_1}$ and $\{Inv\} \boxed{WH \dots} \{ExC\}$. Now
continue with $\boxed{\mathcal{I}_2}$.

Does $Post\ s$ follow from $Inv\ s \wedge \neg s\ sum \leq s\ a$?

Taking Care of *Post*

We have shown $\boxed{\mathcal{I}_1}$ and $\{Inv\} \boxed{WH \dots} \{Exc\}$. Now continue with $\boxed{\mathcal{I}_2}$.

Does *Post* *s* follow from *Inv* *s* $\wedge \neg s \text{ sum} \leq s a$?

Yes!

$(s i)^2 \leq (s a)$ follows from (4)

$(s a) < (s i + 1)^2$ follows from $\neg s \text{ sum} \leq (s a)$ and (2).

The Final Missing Part

\mathcal{I}_3 remains to be shown, i.e.

$$\forall s. PW\ s \rightarrow Inv\ s$$

or, expanding the solutions for PW and Inv

$$\begin{aligned} \forall s. \quad & s\ i = 0 \wedge s\ sum = 1 \wedge s\ tm = 1 \rightarrow \\ & (s\ i + 1)^2 = s\ sum \wedge \\ & s\ tm = (2 * (s\ i)) + 1 \wedge \\ & (s\ i)^2 \leq (s\ a) \end{aligned}$$

This is easy to check.

An Alternative for Tackling the Loop Part

Recall that our loop invariant was “too strong”. An alternative:

$$\frac{\{\lambda s. Inv\ s \wedge s\ sum \leq s\ a\} \boxed{\text{"body"}} \{Inv\}}{\{Inv\} \boxed{WH \dots} \{ExC\}} \textit{While}$$

An Alternative for Tackling the Loop Part

Recall that our loop invariant was “too strong”. An alternative:

$$\begin{array}{c}
 \forall s. (Inv\ s \wedge \\
 s\ sum \leq s\ a) \rightarrow \\
 Inv'\ s
 \end{array}
 \quad
 \frac{}{\{Inv'\} \boxed{\text{"body"}} \{Inv\}}
 \quad
 \text{conseq}$$

$$\frac{\{ \lambda s. Inv\ s \wedge s\ sum \leq s\ a \} \boxed{\text{"body"}} \{Inv\}}{\{Inv\} \boxed{WH \dots} \{ExC\}}
 \quad
 \text{While}$$

An Alternative for Tackling the Loop Part

Recall that our loop invariant was “too strong”. An alternative:

$$\begin{array}{c}
 \{Inv'\}i ::= \lambda s. s\ i + 1\{P'\} \\
 \{P'\}tm ::= \lambda s. s\ tm + 2\{P''\} \\
 \{P''\}sum ::= \lambda s. s\ tm + s\ sum\{Inv\} \\
 \hline
 \forall s. (Inv\ s \wedge s\ sum \leq s\ a) \rightarrow \\
 Inv'\ s \\
 \hline
 \{Inv'\} \boxed{\text{"body"}} \{Inv\} \quad \text{semi}^2 \\
 \hline
 \{ \lambda s. Inv\ s \wedge s\ sum \leq s\ a \} \boxed{\text{"body"}} \{Inv\} \quad \text{conseq} \\
 \hline
 \{Inv\} \boxed{WH \dots} \{ExC\} \quad \text{While}
 \end{array}$$

Alternative (Cont.)

Applying *ass* as before gives

$$\begin{aligned} Inv' = \lambda s. Inv(s[i ::= s\ i + 1][tm ::= s\ tm + 2] \\ [sum ::= s\ tm + 2 + s\ sum]) \end{aligned}$$

We are left with the proof obligation

$$\begin{aligned} \forall s. (Inv\ s \wedge s\ sum \leq s\ a) \rightarrow \\ Inv(s[i ::= s\ i + 1][tm ::= s\ tm + 2] \\ [sum ::= s\ tm + 2 + s\ sum]) \end{aligned}$$

Automating Hoare Proofs

In the **example**, we have verified a program computing the square root.

But this was tedious, and parts of the task can be automated.

Weakest Preconditions

Observation: the Hoare relation is deterministic to a certain extent.

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Idea: we use this fact for the generation of **weakest preconditions**.

Weakest preconditions are:

constdefs $wp :: com \Rightarrow assn \Rightarrow assn$

" $wp\ c\ Q \equiv (\lambda s. \forall t. (s, t) \in C(c) \longrightarrow Q\ t)$ "

So $wp\ c\ Q$ returns the **set of states** containing all states s such that if t is reached from s via c , then the post-condition Q holds for t . Computable?

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So $wp\ c\ Q$ returns the **set of states** containing all states s such that if t is reached from s via c , then the post-condition Q holds for t . Computable? Not obvious.

Equivalence Proofs

Main results of the wp-generator are:

$$\begin{aligned}
 \text{wp_SKIP:} & \quad \text{wp SKIP } Q = Q \\
 \text{wp_Ass:} & \quad \text{wp}(x ::= a) Q = (\lambda s. Q (s[x::=a s])) \\
 \text{wp_Semi:} & \quad \text{wp}(c; d) Q = \text{wp } c (\text{wp } d Q) \\
 \text{wp_If:} & \quad \text{wp (IF } b \text{ THEN } c \text{ ELSE } d) Q = \\
 & \quad (\lambda s. (b s \longrightarrow \text{wp } c Q s) \wedge (\neg b s \longrightarrow \text{wp } d Q s)) \\
 \text{wp_While_True:} & \quad b s \implies \text{wp (WHILE } b \text{ DO } c) Q s = \\
 & \quad \text{wp (c; WHILE } b \text{ DO } c) Q s \\
 \text{wp_While_False:} & \quad \neg b s \implies \text{wp (WHILE } b \text{ DO } c) Q s = Q s \\
 \text{wp_While_if:} & \quad \text{wp (WHILE } b \text{ DO } c) Q s = \\
 & \quad (\text{if } b s \text{ then wp}(c; \text{ WHILE } b \text{ DO } c) Q s \text{ else } Q s)
 \end{aligned}$$

Computing Weakest Preconditions

Except for termination problem due to While, weakest precondition wp can be computed.

This fact can be used for further proof support by verification condition generation.

Idea: for all statements, the **exact** wp is computed, except for the creative step at While, where the assertion INV provided by the user for the invariant is taken.

Computing Weakest Preconditions

Except for termination problem due to While, weakest precondition wp can be computed.

This fact can be used for further proof support by verification condition generation.

Idea: for all statements, the **exact** wp is computed, except for the creative step at While, where the assertion INV provided by the user for the invariant is taken. An additional function vc (“verification condition”) establishes necessary conditions that INV is indeed an invariant.

Annotated IMP Programs

We enrich the syntax by loop-invariants:

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datatype acom =

 Askip

| Aass loc aexp

| Asemi acom acom

| Aif bexp acom acom

| Awhile bexp assn acom

Computing a “Approximative” Weakest Precondition

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primrec

$$\text{awp } \text{Askip } Q = Q$$

$$\text{awp } (\text{Aass } x \ a) \ Q = (\lambda s. \ Q(s[x::=a \ s]))$$

$$\text{awp } (\text{Asemi } c \ d) \ Q = \text{awp } c \ (\text{awp } d \ Q)$$

$$\text{awp } (\text{Aif } b \ c \ d) \ Q = (\lambda s. \ (b \ s \longrightarrow \text{awp } c \ Q \ s) \wedge (\neg b \ s \longrightarrow \text{awp } d \ Q \ s))$$

$$\text{awp } (\text{Awhile } b \ \text{Inv } \ c) \ Q = \text{Inv}$$

Note that awp is not necessarily a wp; this depends if Inv is indeed an invariant.

Verification Condition Generation

Inv is an Invariant if **verification condition**vc holds:

primrec

$$\text{vc Askip } Q = (\lambda s. \text{True})$$

$$\text{vc (Aass } x \ a) \ Q = (\lambda s. \text{True})$$

$$\text{vc (Asemi } c \ d) \ Q = (\lambda s. \text{vc } c \ (\text{awp } d \ Q) \ s \wedge \text{vc } d \ Q \ s)$$

$$\text{vc (Aif } b \ c \ d) \ Q = (\lambda s. \text{vc } c \ Q \ s \wedge \text{vc } d \ Q \ s)$$

$$\begin{aligned} \text{vc (Awhile } b \ \text{Inv } \ c) \ Q = & (\lambda s. (\text{Inv } s \wedge \neg b \ s \longrightarrow Q \ s) \wedge \\ & (\text{Inv } s \wedge b \ s \longrightarrow \text{awp } c \ \text{Inv } s) \wedge \\ & \text{vc } c \ \text{Inv } s) \end{aligned}$$

Results on vc (1)

The following facts on vc and awp makes this concept powerful:

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Theorem 6 (Soundness (vc_sound)):

$$(\forall s. \text{vc } ac \ Q \ s) \implies \vdash \{ \text{awp } ac \ Q \} \text{astrip } ac \ \{ Q \}$$

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Theorem 6 (Soundness (vc_sound)):

$$(\forall s. \text{vc } ac \ Q \ s) \implies \vdash \{ \text{awp } ac \ Q \} \text{astrip } ac \ \{ Q \}$$

vc generated from the annotated program holds, then there exists a Hoare-proof for the program without annotations showing that the postcondition follows from its weakest (annotated) precondition.

Results on vc (2)

Moreover, we have:

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Theorem 7 (Completeness (vc_complete)):

$$\begin{aligned} \vdash \{P\} c \{Q\} \implies \exists ac. \quad & \text{astrip } ac = c \wedge \\ & (\forall s. \text{vc } ac \ Q \ s) \wedge \\ & (\forall s. P \ s \longrightarrow \text{awp } ac \ Q \ s) \end{aligned}$$

Results on vc (2)

Moreover, we have:

Theorem 7 (Completeness (vc_complete)):

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If a Hoare-proof exists, there must exist an annotated program, for which vc generates a true formula and whose precondition P implies the its weakest precondition.

Quintessence: vc abstracts Hoare-proofs away from (imperative) program verification. . .

Summary

- IMP closely follows the standard textbook [Win96].

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- IMP closely follows the standard textbook [Win96].
- Isabelle/HOL is a powerful framework for embedding imperative languages.
- Isabelle/HOL is also a framework for state-of-the-art languages like JAVA.
- Even verification condition generators can be proven sound and complete **within** HOL.

More Detailed Explanations

Equivalence Proofs

Summarizing, we have the following equivalence results:

- natural vs. transition semantics
- denotational vs. natural semantics

Locations

We realize program variables via pointers (locations). The type of pointers is an **abstract datatype**.

Defining of **values** by **nat** is just a simplification.

A **state** is a function taking a location to a value, i.e. intuitively, each **program variable** corresponds to a location, each access to a program variable is an application of state to the location of this variable.

The Intuition of Natural Semantics

The idea of the natural semantics is that a program relates two states, the “input state” and the “output state”, provided that it terminates.

This is similar to denotational at first sight, but the treatment of “recursive” constructs such as the `WHILE` is different: denotational semantics reduces these to fixpoint operators, natural semantics to the (meta)-question, if a derivation for the transition is possible or not.

The Intuition of Transition Semantics

Unlike the *natural semantics*, the transition semantics records the single steps of the computation. A *configuration* is a pair consisting of a program and a state, and one step reaches a new program and a new state.

The intuition behind the program-component is “the program to be executed”; it is manipulated in a stack-like manner during the evaluation of the rules.

An in-depth investigation of the rules reveals, that there are only finitely many different programs-components in all configuration traces; these correspond to “positions” in a program. (Note that assignments have a “position before” and a “position after” execution). Thus, this component can be seen as a program counter (PC).

Understanding Gamma

We discuss the approximation relation Gamma in more detail:

"Gamma b cd $\equiv (\lambda \text{phi.} \{(s,t). (s,t) : (\text{phi } \text{O } \text{cd}) \wedge b(s)\} \cup \{(s,t). s=t \wedge \neg b(s)\})$ "

Note that in the definition of WHILE , the second argument cd to Gamma is used for the meaning of the **body** of the loop.

Thus, the underlying principle is similar to the one used when defining the transitive closure by lfp .

Let

$$S_0 = \{(s, t) \mid s = t \wedge \neg b(s)\}$$

be the initial approximation (the subset of the identity relation, for which the condition b is false).

Then we can iterate from S_0 via composition cd^n arbitrarily many

transitions through the body of the loop. The lfp represents the limit of this approximation process.

Note that for $b = \lambda x. \text{ true}$ S_0 is empty, therefore $\text{Gamma } b \text{ cd}$ is empty and, consequently, the denotational semantics C will yield the empty relation in such cases.

A Table of Values

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i counts the number of times the loop is entered, i.e. the final value of i is the number of times the loop was entered. This number depends on a . The following table shows that final values of i , tm and sum depending on the value of a :

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	i	tm	sum
$0 \leq a < 1$	0	1	1
$1 \leq a < 4$	1	3	4
$4 \leq a < 9$	2	5	9
$9 \leq a < 16$	3	7	16
$16 \leq a < 25$	4	9	25
$25 \leq a < 36$	5	11	36
$36 \leq a < 49$	6	13	49

sum takes the values of all squares successively, computed by the famous binomial formula:

$$(i + 1)^2 = i^2 + 2i + 1$$

Since tm takes the value $2i + 1$ for all i successively, it follows that

$sum + tm$ always gives the next value of sum .

$(s\ i)$, $(s\ a)$ **etc.**

Informally we talk about variables i , x etc. and say “ x has value 5”, for example. But formally, program variables are realized via **locations**, and when accessing a program variable, we get expressions of the form $s\ x$. That is, $s\ x$ is the value of variable x .

Nondeterminacy in the Hoare Calculus

The *conseq* rule can always be applied. For all other commands, the choice for a hoare-rule is uniquely determined.

Relative Completeness

After Gödel, logicians tend to be nervous whenever a logic is claimed to be complete, in particular if arithmetic is involved as is the case for IMP. Relative completeness means that for any valid Hoare triple in validity relation there will be a proof in hoare logic.

However, will we be able to derive in HOL that any semantically valid hoare triple is valid in the sense of validity relation? What if we write in a precondition something like “if Goldbach’s conjecture holds” or, worse, “if we can solve an arbitrary diophantine equation”?

The answer is **no** since HOL itself is incomplete wrt. standard models.

Program “Fragment”

This is the entire program, namely:

```
tm   ::= λs. 1;
sum  ::= λs. 1;
i    ::= λs. 0;
WHILE λs. s sum <= s a DO
  (i   ::= λs. s i + 1;
   tm  ::= λs. s tm + 2;
   sum ::= λs. s tm + s sum)
```

(return to main proof tree)

Program Fragment

This is the program fragment starting from $sum ::=$, namely:

```
sum ::=  $\lambda s. 1$ ;
```

```
i ::=  $\lambda s. 0$ ;
```

```
WHILE  $\lambda s. s \text{ sum} \leq s \text{ a}$  DO
```

```
  (i ::=  $\lambda s. s \text{ i} + 1$ ;
```

```
   tm ::=  $\lambda s. s \text{ tm} + 2$ ;
```

```
   sum ::=  $\lambda s. s \text{ tm} + s \text{ sum}$ )
```

(return to main proof tree)

Program Fragment

This is the program fragment starting from $i ::=$, namely:

```
i ::=  $\lambda x. 0$ ;  
WHILE  $\lambda s. s \text{ sum} \leq s a$  DO  
  ( $i ::= \lambda s. s i + 1$ ;  
    $tm ::= \lambda s. s tm + 2$ ;  
    $sum ::= \lambda s. s tm + s \text{ sum}$ )
```

(return to main proof tree)

Invariant

$$Inv \equiv \text{''}\lambda s.(s\ i + 1)^2 = s\ sum \wedge s\ tm = (2 * s\ i) + 1 \wedge s\ i^2 \leq a\text{''}$$

Program Fragment

This is the program fragment starting from *WHILE*, namely:

```
WHILE  $\lambda s. s \text{ sum} \leq s \text{ a}$  DO  
  ( $i ::= \lambda s. s \text{ i} + 1$ ;  
    $\text{tm} ::= \lambda s. s \text{ tm} + 2$ ;  
    $\text{sum} ::= \lambda s. s \text{ tm} + s \text{ sum}$ )
```

(return to main proof tree)

Program Fragment

This is the program fragment consisting of the loop body, namely:

```
i      ::= λs. s i + 1;  
tm     ::= λs. s tm + 2;  
sum    ::= λs. s tm + s sum  
(return to main proof tree)
```


A Missing Part

$\boxed{\mathcal{I}_1}$ is the formula

$$\forall s. Pre\ s \rightarrow PW(s["i, sum, tm"])$$

where Pre is defined above and PW is a metavariable (“precondition of *WHILE*”).

(return to main proof tree)

A Missing Part

$\boxed{\mathcal{I}_2}$ is the formula

$$\forall s. ExC\ s \rightarrow Post\ s,$$

i.e.

$$\forall s. Inv\ s \wedge \neg sum\ s \leq\ s\ a \rightarrow Post\ s,$$

where $Post$ is defined above and Inv is a metavariable (“loop invariant”).
(return to main proof tree)

A Missing Part

$\boxed{A_1}$ is the proof tree

$$\frac{\{\lambda s.PW(s["i, sum, tm"])\}tm ::= \lambda x.1\{\lambda s.PW(s["i, sum"])\}}{ass}$$

where PW is a metavariable (“precondition of *WHILE*”).

(return to main proof tree)

A Missing Part

$\boxed{A_2}$ is the proof tree

$$\frac{\{ \lambda s. PW(s[i ::= 0][sum ::= 1]) \} sum ::= \lambda x. 1 \{ \lambda s. PW(s[i ::= 0]) \}}{ass}$$

where PW is a metavariable (“precondition of *WHILE*”).

(return to main proof tree)

A Missing Part

A_3 is the proof tree

$$\frac{}{\{\lambda s.PW(s[i ::= 0])\}i ::= \lambda x.0\{PW\}} \text{ass}$$

where PW is a metavariable (“precondition of *WHILE*”).
(return to main proof tree)

A Missing Part

$\boxed{\mathcal{I}_3}$ is the formula

$$\forall s. PW\ s \rightarrow Inv\ s$$

where PW is a metavariable (“precondition of *WHILE*”) and Inv is a metavariable (“loop invariant”).

(return to main proof tree)

A Missing Part

$\boxed{\mathcal{I}_4}$ is the formula

$$\forall s. PW\ s \rightarrow PW\ s$$

which is of course trivial to prove.

(return to main proof tree)

An Abbreviation for an Updated State

We use $s["i, sum, tm"]$ as abbreviation for

$$s[i ::= 0][sum ::= 1][tm ::= 1]$$

(return to main proof tree)

An Abbreviation for an Updated State

We use $s["i, sum"]$ as abbreviation for

$$s[i ::= 0][sum ::= 1]$$

(return to main proof tree)

An Abbreviation for an Updated State

We use $s[\"i\"]$ as abbreviation for

$$s[i ::= 0]$$

(return to main proof tree)

What must *Inv* Be?

Recall that we had to prove the three formulas

$$\begin{aligned} \{\lambda s. Inv\ s \wedge s\ sum \leq s\ a\}i &::= \lambda s. s\ i + 1\{P'\} \\ \{P'\}tm &::= \lambda s. s\ tm + 2\{P''\} \\ \{P''\}sum &::= \lambda s. s\ tm + s\ sum\{Inv\} \end{aligned}$$

all by *ass*. Dealing with the second and third formula using *ass*, we found that

$$P' = \lambda s'. Inv(s'[tm ::= s' tm + 2][sum ::= s' tm + 2 + s' sum]).$$

Therefore, to show

$$\{\lambda s. Inv\ s \wedge s\ sum \leq s\ a\}i ::= \lambda s. s\ i + 1\{P'\}$$

as well, Inv must have such a form that the formula becomes an instance of ass .

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